# Sparse interpolation, exponential analysis, Padé approximation, tensor decomposition

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# Motivation

# Motivation

- explosives identification
- induction motor current signature analysis
- magnetic resonance spectroscopy / imaging
- fluorescence lifetime imaging
- DOA/AOA problems
- transient detection
- big data analytics
- speech/music signal processing
- ...

# Sparse interpolation

# Sparse interpolation

interpolate

$$f(x) = \alpha_1 + \alpha_2 x^{100}$$

- Newton/Lagrange interpolation: 101 samples
- only 4 unknowns: $\alpha_1$, $\alpha_2$, $x^0$, $x^{100}$!
- how to solve it from 4 samples?

# Sparse interpolation

$$\sum_{i=1}^{n} \alpha_i x_s^{k_i} = f_s, \quad n \ll \max(k_i), \quad k_i \in \mathbb{N}$$

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$$\sum_{i=1}^{n_1} \alpha_{i,1} \cos(\phi_{i,1} x_s) + \sum_{i=1}^{n_2} \alpha_{i,2} \sin(\phi_{i,2} x_s) = f_s$$

# Sparse interpolation

$$\sum_{i=1}^{n} \alpha_i x_s^{k_i} = f_s, \quad n \ll \max(k_i), \quad k_i \in \mathbb{N}$$

$$\sum_{i=1}^{n_1} \alpha_{i,1} \cos(\phi_{i,1} x_s) + \sum_{i=1}^{n_2} \alpha_{i,2} \sin(\phi_{i,2} x_s) = f_s$$

$$x_s = s\Delta, \quad s = 0, 1, 2, \ldots$$

$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = f_s, \quad \phi_i \in \mathbb{C}$$

# Exponential analysis

$$\phi(x) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i x) + \varepsilon(x)$$

$$\alpha_i \in \mathbb{C}$$

$$\phi_i = \beta_i + \mathrm{i}\gamma_i, \qquad |\Im(\phi_i)| < M/2, \qquad \Delta = 2\pi/M$$

equidistant $x_s = s\Delta, \quad s = 0, 1, 2, \ldots, 2n-1, \ldots$

$$f_s := \phi(x_s), \qquad \varepsilon_s := \varepsilon(x_s)$$

- wide spectrum
- high resolution
- small SNR

# Exponential analysis

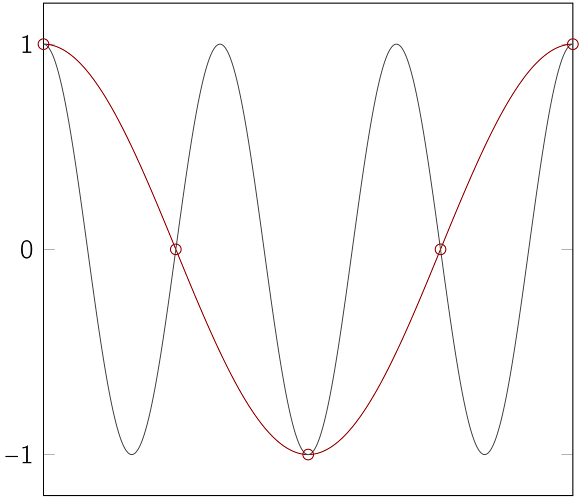



Figure: Too few interpolation points introduce aliasing

# Sparse interpolation

interpolation problem:

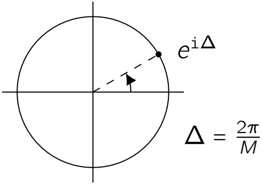
$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = f_s, \quad s = 0, \ldots, 2n-1$$

$$x_s = s\Delta, \quad \Delta = 2\pi/M$$

$$|\Im(\phi_i)| < M/2, \quad \Phi_i = \exp(\phi_i \Delta),$$

$$f_s = \sum_{i=1}^{n} \alpha_i \Phi_i^s, \quad s = 0, \ldots, 2n-1$$

$$\begin{cases} \alpha_1 + \cdots + \alpha_n = f_0 \\ \alpha_1 \Phi_1 + \cdots + \alpha_n \Phi_n = f_1 \\ \quad \vdots \\ \alpha_1 \Phi_1^{2n-1} + \cdots + \alpha_n \Phi_n^{2n-1} = f_{2n-1} \end{cases}$$



# Sparse interpolation

finding $\Phi_i$:

$$\prod_{i=1}^{n}(z-\Phi_i) = z^n + d_{n-1}z^{n-1} + \cdots + d_1 z + d_0$$

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \alpha_i \Phi_i^s (\Phi_i^n + d_{n-1}\Phi_i^{n-1} + \cdots + d_0) \\ &= \sum_{i=1}^{n} \alpha_i \Phi_i^{n+s} + \sum_{j=0}^{n-1} d_j \left( \sum_{i=1}^{n} \alpha_i \Phi_i^{j+s} \right) \\ &= f_{s+n} + \sum_{j=0}^{n-1} d_j f_{s+j} \end{aligned}$$

# Sparse interpolation

$$\begin{pmatrix} f_0 & \dots & f_{n-1} \\ \vdots & \iddots & \vdots \\ f_{n-1} & \dots & f_{2n-2} \end{pmatrix} \begin{pmatrix} d_0 \\ \vdots \\ d_{n-1} \end{pmatrix} = - \begin{pmatrix} f_n \\ \vdots \\ f_{2n-1} \end{pmatrix}$$

Hankel matrix:

$$H_n^{(r)} = \begin{pmatrix} f_r & \dots & f_{r+n-1} \\ \vdots & \iddots & \vdots \\ f_{r+n-1} & \dots & f_{r+2n-2} \end{pmatrix}$$

# Sparse interpolation

Hadamard polynomial:

$$H_n^{(0)}(z) = \begin{vmatrix} f_0 & \dots & f_{n-1} & f_n \\ \vdots & \iddots & \vdots & \vdots \\ f_{n-1} & \dots & f_{2n-2} & f_{2n-1} \\ 1 & \dots & z^{n-1} & z^n \end{vmatrix}$$

$$\prod_{i=1}^{n}(z - \Phi_i) = \frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|}$$
$$= z^n + d_{n-1}z^{n-1} + \dots + d_1 z + d_0$$

# Sparse interpolation

formally orthogonal polynomial:

$$\gamma : z^s \to f_s, \quad s = 0, 1, \ldots$$

$$\frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|} \perp_\gamma z^i, \quad i = 0, \ldots, n-1$$

$$\gamma\left(z^i \frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|}\right) = 0, \quad i = 0, \ldots, n-1$$

[Henrici, 1974]

# Sparse interpolation

roots of $\dfrac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|}$ from GEP:

$$H_n^{(0)} = \begin{pmatrix} 1 & \dots & & 1 \\ \Phi_1 & \Phi_2 & \dots & \Phi_n \\ \vdots & & & \vdots \\ \Phi_1^{n-1} & \dots & & \Phi_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \Phi_1 & \dots & \Phi_1^{n-1} \\ \vdots & \Phi_2 & & \vdots \\ & \vdots & & \vdots \\ 1 & \Phi_n & \dots & \Phi_n^{n-1} \end{pmatrix}$$

$$= V_n^T D_\alpha V_n$$

$$H_n^{(1)} = V_n^T D_\alpha \begin{pmatrix} \Phi_1 & & \\ & \ddots & \\ & & \Phi_n \end{pmatrix} V_n$$

# Sparse interpolation

$$\det\left(H_n^{(1)} - \lambda H_n^{(0)}\right) = \det\left( V_n^T D_\alpha \begin{pmatrix} \Phi_1 - \lambda & & \\ & \ddots & \\ & & \Phi_n - \lambda \end{pmatrix} V_n \right)$$
$$= 0 \text{ for } \lambda = \Phi_i, \quad i = 1, \ldots, n$$

[Hua and Sarkar, 1990]

# Sparse interpolation

finding $\phi_i$:

$$\exp(\phi_i) = \exp(\Re(\phi_i))e^{\mathrm{i}\Im(\phi_i)}$$

$$|\Im(\phi_i)| < \frac{M}{2}:$$

$$\begin{aligned}\arg(\Phi_i) &= \arg(\exp(\phi_i \Delta)) \\ &= \Im(\phi_i)\,\frac{2\pi}{M} \in \,]-\pi, \pi[\end{aligned}$$

# Sparse interpolation

finding $\alpha_i$:

$$\sum_{i=1}^{n} \alpha_i \Phi_i^{s+j} = f_{s+j}, \quad s = 0, \dots, n-1, \quad 0 \le j \le n$$

$$\begin{pmatrix} \Phi_1^j & \dots & \Phi_n^j \\ \vdots & & \vdots \\ \Phi_1^{j+n-1} & \dots & \Phi_n^{j+n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_j \\ \vdots \\ f_{j+n-1} \end{pmatrix}$$

remaining interpolation conditions are linearly dependent

# Sparse interpolation

finding $n$:

$$N < n : \left|H_N^{(r)}\right| \not\equiv 0, \quad r = 0, 1, \ldots$$

$$N = n : \left|H_N^{(r)}\right| \neq 0 \quad \text{if } \Phi_i \neq \Phi_j \text{ for } i \neq j$$ [Kaltofen and Lee, 2003]

$$N > n : \left|H_N^{(r)}\right| \equiv 0, \quad r = 0, 1, \ldots$$

# Exponential analysis

# Exponential analysis

$$\phi(x) = \sum_{i=1}^{4} \alpha_i \exp(\phi_i x)$$

$$\alpha_1 = 1 \qquad \phi_1 = 0$$
$$\alpha_2 = 2.4 \qquad \phi_2 = -5 + 19.97\mathtt{i}$$
$$\alpha_3 = -2.1 \qquad \phi_3 = 3 + 45\mathtt{i}$$
$$\alpha_4 = 0.2 \qquad \phi_4 = 5.3\mathtt{i}$$

evaluate at $x_s = s\frac{2\pi}{100}$, $\quad M = 100$, $\quad |\Im(\phi_i)| < 50$

sequence $f_0, \ldots, f_7, \ldots$ is linearly generated

# Exponential analysis

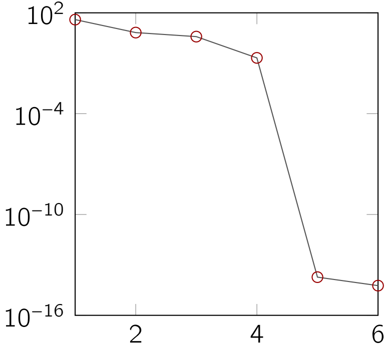



Figure: $H_N^{(0)}$ singular, $N = 6$

# Exponential analysis

$$\phi(x) = 174.13 \exp(-x/22) + 19.348 \exp(-x/80) + \varepsilon(x)$$

$$10 \log_{10} \left( \frac{\sum_{s=0}^{255} \phi^2(x_s)}{\sum_{s=0}^{255} \varepsilon^2(x_s)} \right) = 34$$

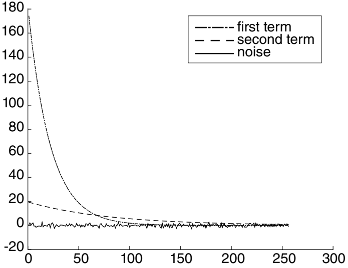



Figure: Individual components including the discrete noise

# Exponential analysis

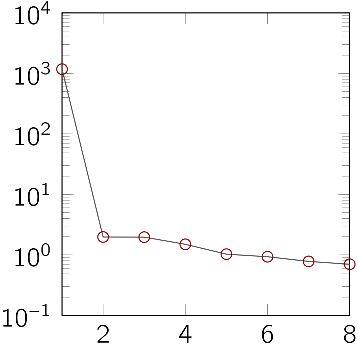



Figure: Log-plot of singular values of $H_8^{(0)}$ with 34 dB white Gaussian noise added

# Approximation theory

# Approximation theory

$$f_s = \sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s), \qquad s = 0, 1, \ldots, 2n-1$$

$$f(z) = \sum_{j=0}^{\infty} f_j z^j, \qquad f_j = 0, \qquad j < 0$$

$$p(z) = \sum_{i=0}^{m} a_i z^i,$$

$$q(z) = \sum_{i=0}^{n} b_i z^i$$

$$\left(\sum_{j=0}^{\infty} f_j z^j\right) q(z) - p(z) = \sum_{i \geq m+n+1} c_i z^i$$

# Approximation theory

$$\begin{cases} f_0 b_0 = a_0 \\ f_1 b_0 + f_0 b_1 = a_1 \\ \quad \vdots \\ f_m b_0 + \cdots + f_{m-n} b_n = a_m \end{cases}$$

$$b_0 = 1$$

$$\begin{cases} f_{m+1} b_0 + \cdots + f_{m-n+1} b_n = 0 \\ \quad \vdots \\ f_{m+n} b_0 + \cdots + f_m b_n = 0 \end{cases}$$

$$H_n^{(m+1-n)} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = - \begin{pmatrix} f_{m+1} \\ \vdots \\ f_{m+n} \end{pmatrix}$$

$$[m/n](z) := p(z)/q(z)$$

# Approximation theory

$$f_s = \sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = \sum_{i=1}^{n} \alpha_i \Phi_i^s$$

$$f(z) = \sum_{j=0}^{\infty} f_j z^j = \sum_{i=1}^{n} \frac{\alpha_i}{1 - z\Phi_i}$$

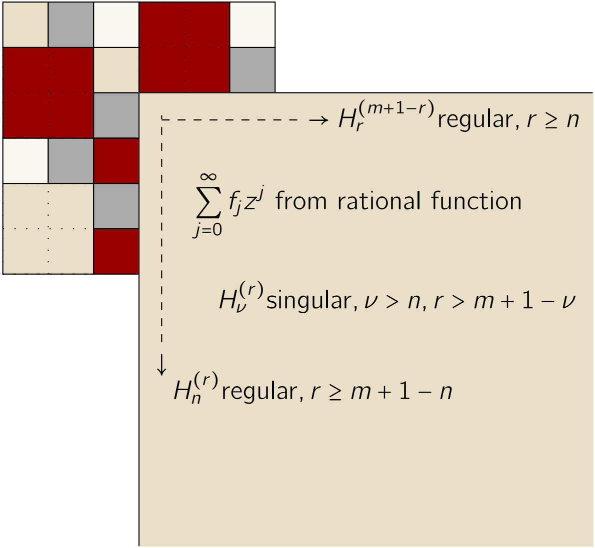
related to Laplace transform of $\sum_{i=1}^{n} \alpha_i \exp(\phi_i x)$

# Approximation theory

$$[n-1/n](z) = p(z)/q(z)$$

$$\begin{aligned} q(z) &= \prod_{i=1}^{n}(1 - z\Phi_i) \\ &= z^n \frac{H_n^{(0)}(1/z)}{\left|H_n^{(0)}\right|} \\ &= b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + 1 \end{aligned}$$

# Approximation theory

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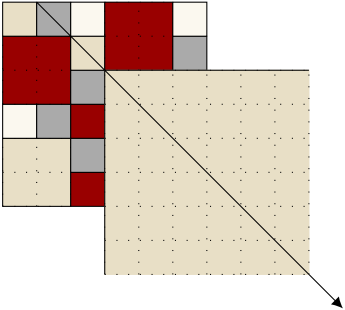
# Approximation theory

$f(z) + \varepsilon(z)$ analytic except for a countable number of poles [Nuttall, 1970] and essential singularities [Pommerenke, 1973]

$$\Downarrow$$

$[\nu - 1/\nu](z) \to f(z) + \varepsilon(z)$ in measure on compact sets, i.e.

$$\Lambda_2\left(\{z : |f(z) + \varepsilon(z) - [\nu - 1/\nu](z)| \geq \tau\}\right) \to 0$$



# Approximation theory

mathematical (noise free):

1. build $H_\nu^{(0)}, \quad \nu = 0, 1, 2, \ldots$
2. $H_\nu^{(0)} = U \Sigma V^T$ singular value decomposition
3. $\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_\nu \end{pmatrix}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > \sigma_{n+1} = \cdots = \sigma_\nu = 0$
4. find $\Phi_i, \phi_i, \alpha_i, i = 1, \ldots, n$ from $H_n^{(1)} v_i = \Phi_i H_n^{(0)} v_i$ and the interpolation conditions

# Approximation theory

numerical (with noise):

1. take $\nu$ large enough so that in the singular value decomposition noise is clearly separated from $n$
2. solve $H_\nu^{(1)} v_i = \lambda_i H_\nu^{(0)} v_i, \quad i = 1, \ldots, \nu, \quad \lambda_i = \Phi_i, \quad i = 1, \ldots, n$
3. find $\phi_i$
4. solve $\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_j) = f_j, \quad 0 \le j \le 2\nu - 1$

# Exponential analysis

# Exponential analysis

$$\phi(x) = \sum_{i=1}^{4} \alpha_i \exp(\phi_i x)$$

$$\alpha_1 = 1 \qquad \phi_1 = 0$$
$$\alpha_2 = 2 \qquad \phi_2 = -0.2 + 39.5\mathtt{i}$$
$$\alpha_3 = 4 \qquad \phi_3 = -0.5 + 40\mathtt{i}$$
$$\alpha_4 = 8 \qquad \phi_4 = -1$$

evaluate at $x_s = s\frac{2\pi}{100}$, $\quad M = 100$, $\quad |\Im(\phi_i)| < 50$

$\|\epsilon(z)\|_\infty = 10^{-2}$, $\quad$ uniform random noise

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# Exponential analysis

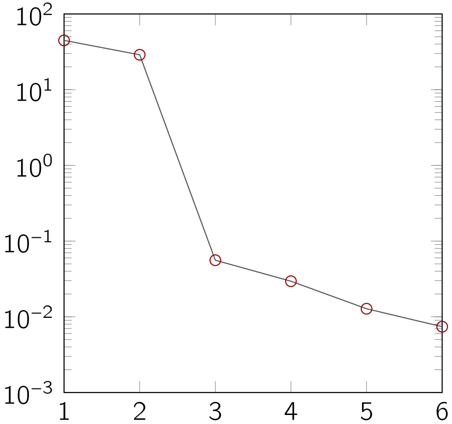


Figure: Singular values $H_\nu^{(0)}$ with $n = 4, \nu = 6$

# Exponential analysis

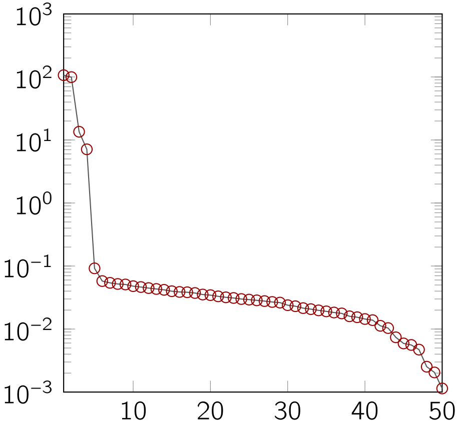


Figure: Singular values $H_\nu^{(0)}$ with $n = 4, \nu = 50$

# Exponential analysis

$$\phi(x) = 174.13 \exp(-x/22) + 19.348 \exp(-x/80) + \varepsilon(x)$$

$$10 \log_{10} \left( \frac{\sum_{s=0}^{255} \phi^2(x_s)}{\sum_{s=0}^{255} \varepsilon^2(x_s)} \right) = 34$$

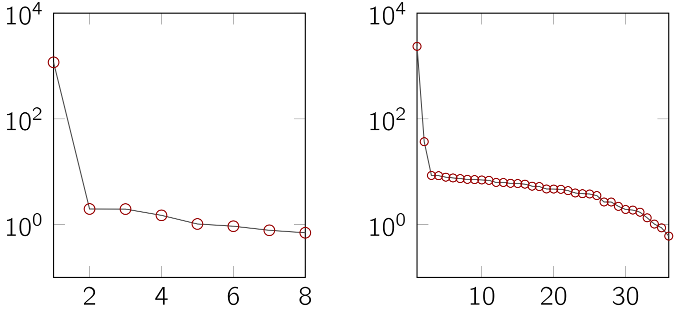


Figure: Log-plot of singular values of $H_8^{(0)}$ and $H_{36}^{(0)}$ with 34 dB white Gaussian noise added

# Tensor decomposition

# Tensor decomposition

with symmetric tensor $[t_{j_1 \ldots j_k}]_{j_1,\ldots,j_k=0}^{d}$
of order $k$ and dimension $d+1$

$$t_{j_1 \ldots j_k} = t_{\pi(j_1)\ldots\pi(j_k)}$$

associate the $(d+1)$-variate homogeneous polynomial of degree $k$

$$\sum_{j_1 \le \cdots \le j_k = 0}^{d} \left( \sum_{\pi} t_{\pi(j_1)\ldots\pi(j_k)} \right) z_{j_1} \ldots z_{j_k}$$

e.g. $k=3,\ d=1$

$$\left(\begin{array}{cc|cc} t_{000} & t_{010} & t_{001} & t_{011} \\ t_{100} & t_{110} & t_{101} & t_{111} \end{array}\right)$$

$$t_{000} z_0^3 + (t_{100} + t_{010} + t_{001}) z_0^2 z_1 + (t_{110} + t_{101} + t_{011}) z_0 z_1^2 + t_{111} z_1^3$$

# Tensor decomposition

compactly written as

$$\sum_{|\kappa|=k} c_\kappa Z^\kappa,$$

$$Z = (z_0, \dots, z_n), \quad \kappa = (k_0, \dots, k_n)$$

$$Z^\kappa = z_0^{k_0} \dots z_n^{k_n}, \quad |\kappa| = k_0 + \cdots + k_n$$

set $z_0 = 1$ and for $d = 1$, $z = z_1$, $k = k_1$:

$$\sum_{|\kappa|=k} c_\kappa Z^\kappa = \sum_{j=0}^{k} c_j z^j$$

# Tensor decomposition

e.g. $k = 3$, $d = 1$

$$t_{000} + (t_{100} + t_{010} + t_{001})z + (t_{110} + t_{101} + t_{011})z^2 + t_{111}z^3$$

tensor decomposition problem

$$\sum_{j=0}^{k} c_j z^j = \sum_{i=1}^{r} w_i (1 + \lambda_i z)^k = \sum_{j=0}^{k} \binom{k}{j} \sum_{i=1}^{r} w_i \lambda_i^j z^j$$

# Tensor decomposition

denote

$$\sigma_s = c_s / \binom{k}{s},$$

then

$$\sum_{i=1}^{r} w_i \lambda_i^s = \sigma_s, \qquad s = 0, \ldots, k$$

# Tensor decomposition

$$\lambda_i = \Delta^{m_i}, \qquad \Delta = \exp(2\pi \mathtt{i}/M), \qquad \frac{M}{2} > \max_{i=1,\dots,r} (\Im(m_i))$$

$$x_s = \Delta^s = \exp(2\pi \mathtt{i} s/M)$$

$$\begin{aligned}\sum_{i=1}^{r} w_i \lambda_i^s &= \sum_{i=1}^{r} w_i \Delta^{m_i s} \\ &= \sum_{i=1}^{r} w_i x_s^{m_i} \\ &= \sigma_s\end{aligned}$$

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# Tensor decomposition

e.g. $k = 3,\ d = 1$

$$\left(\begin{array}{cc|cc} t_{000} & t_{010} & t_{001} & t_{011} \\ t_{100} & t_{110} & t_{101} & t_{111} \end{array}\right) = \left(\begin{array}{cc|cc} 4 & 3 & 3 & 6 \\ 3 & 6 & 6 & 17 \end{array}\right)$$

$$\sum_{j=0}^{3} c_j z^j = 4 + 9z + 18z^2 + 17z^3$$

$$\sigma_0 = c_0 = 4$$
$$\sigma_1 = c_1/3 = 3$$
$$\sigma_2 = c_2/3 = 6$$
$$\sigma_3 = c_3 = 17$$

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# Tensor decomposition

$$H_2^{(0)} = \begin{pmatrix} \sigma_0 & \sigma_1 \\ \sigma_1 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 6 \end{pmatrix},$$

$$H_2^{(1)} = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 17 \end{pmatrix}$$

$$\sum_{j=0}^{3} c_j z^j = \frac{5}{8}(1+3z)^3 + \frac{27}{8}(1+\frac{1}{3}z)^3 = \sum_{i=1}^{r} w_i (1+\lambda_i z)^3$$

$$\left| H_3^{(0)} \right| = 0 \Rightarrow r = 2$$

[Brachat et al., 2010]

# Tensor decomposition

Hadamard polynomial:

$$\begin{vmatrix} 4 & 3 & 6 \\ 3 & 6 & 17 \\ 1 & z & z^2 \end{vmatrix} / \left| H_2^{(0)} \right| = z^2 - \frac{10}{3} z + 1$$

roots: $\lambda_1 = 3,\ \lambda_2 = 1/3$
equals reverse of Padé denominator of approximant [1/2] for

$$\sigma_0 + \sigma_1 z + \sigma_2 z^2 + \sigma_3 z^3 + \dots$$

or

$$4 + 3z + 6z^2 + 17z^3 + \dots$$

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